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Edge Universality for Orthogonal Ensembles of Random Matrices

M. Shcherbina

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Abstract We prove edge universality of local eigenvalue statistics for orthogonal invariant matrix models with real analytic potentials and one interval limiting spectrum. Our starting point is the result of Shcherbina (Commun. Math. Phys. 285, 957–974, 2009) on the representation of the reproducing matrix kernels of orthogonal ensembles in terms of scalar reproducing kernel of corresponding unitary ensemble.

Keywords Random matrices · Orthogonal ensembles · Universality

1 Introduction and Main Results

We study ensembles of $n \times n$ real symmetric (or Hermitian) matrices M with the probability distribution

$$P_n(M)dM = Z_{n,\beta}^{-1} \exp\left\{-\frac{n\beta}{2}\mathrm{Tr}\, V(M)\right\} dM, \tag{1.1}$$

where $Z_{n,\beta}$ is a normalization constant, $V : \mathbb{R} \to \mathbb{R}_+$ is a Hölder function satisfying the condition

$$|V(\lambda)| \geq 2(1+\epsilon)\log(1+|\lambda|). \tag{1.2}$$

A positive parameter β here assumes the values $\beta = 1$ (in the case of real symmetric matrices) or $\beta = 2$ (in the Hermitian case), and dM means the Lebesgue measure on the algebraically independent entries of M.

The joint eigenvalue distribution corresponding to (1.1) has the form (see [13])

$$p_{n,\beta}(\lambda_1, \ldots, \lambda_n) = Q_{n,\beta}^{-1} \prod_{i=1}^{n} e^{-n\beta V(\lambda_i)/2} \prod_{1 \leq j < k \leq n} |\lambda_i - \lambda_j|^{\beta}, \tag{1.3}$$

M. Shcherbina (✉)
Institute for Low Temperature Physics, 47 Lenin ave., Kharkov, Ukraine
e-mail: shcherbi@ilt.kharkov.ua

where $Q_{n,\beta}$ is a normalization constant. For both cases ($\beta = 1, 2$) the behavior of Normalized Counting Measure (NCM) of eigenvalues is now well understood. According to [3, 11], NCM converges weakly in probability to the non random limiting measure $\mathcal{N}$ known as Integrated Density of States (IDS) of the ensemble. The IDS is absolutely continuous, if V' satisfies the Lipshitz condition [17]. The non-negative density $\rho(\lambda)$ is called Density of States (DOS) of the ensemble. IDS can be found as a unique solution of a certain variational problem (see [3, 11, 17]).

To study the local regimes for ensembles (1.1) means to study the behavior of marginal densities

$$p^{(n)}_{l,\beta}(\lambda_1, \dots, \lambda_l) = \int_{\mathbb{R}^{n-l}} p_{n,\beta}(\lambda_1, \dots, \lambda_l, \lambda_{l+1}, \dots, \lambda_n) d\lambda_{l+1} \dots d\lambda_n \tag{1.4}$$

in the scaling limit, when $\lambda_i = \lambda_0 + x_i/n^\kappa$ $(i = 1, \dots, l)$, and κ is a constant, depending on the behavior of DOS $\rho(\lambda)$ in a small neighborhood of λ_0. If $\rho(\lambda_0) \neq 0$, then $\kappa = 1$, if $\rho(\lambda_0) = 0$ and $\rho(\lambda) \sim |\lambda - \lambda_0|^\alpha$, then $\kappa = 1/(1 + \alpha)$. The universality conjecture states that the scaling limits of all marginal densities are universal, i.e. do not depend on V and depend only on α and β. One of the most known quantity probing the local regime is the gap probability, i.e., the probability that there is no eigenvalues in the interval $\Delta_n(a, b) = [\lambda_0 + a/n^\kappa, \lambda_0 + b/n^\kappa]$

$$E_{n,\beta}(\Delta_n(a, b)) = \mathbf{E}\left\{ \prod_{k=1}^{k} (1 - \mathbf{1}_{\Delta_n(a,b)}(\lambda_i)) \right\}. \tag{1.5}$$

Thus, results on the universality of local eigenvalue statistics usually include proofs of universality of the gap probability.

For unitary ensembles all marginal densities can be represented (see [13]) in terms of so called reproducing kernel

$$K_n(\lambda, \mu) = \sum_{l=0}^{n-1} \psi^{(n)}_l(\lambda)\psi^{(n)}_l(\mu), \tag{1.6}$$

where

$$\psi^{(n)}_l(\lambda) = \exp\{-nV(\lambda)/2\} p^{(n)}_l(\lambda), \quad l = 0, \dots, \tag{1.7}$$

and $\{p^{(n)}_l\}_{l=0}^n$ are orthogonal polynomials on $\mathbb{R}$ associated with the weight $w_n(\lambda) = e^{-nV(\lambda)}$, i.e.,

$$\int p^{(n)}_l(\lambda) p^{(n)}_m(\lambda) w_n(\lambda) d\lambda = \delta_{l,m}. \tag{1.8}$$

In particular,

$$E_{n,2}(\Delta_n(a, b)) = \det\{1 - K_{\Delta_n(a,b)}\},$$

where $\det\{\dots\}$ is the Fredholm determinant and $K_{\Delta_n(a,b)}$ is the integral operator with the kernel (1.6) in $L^2(\Delta_n(a, b))$. Hence, the problem to study marginal distributions is replaced by the problem to study the reproducing kernel K_n in the scaling limit.

The problem was solved in many cases. For example, in the bulk case ($\rho(\lambda_0) \neq 0$) it was shown in [14] (see also [16]) that for a general class of V (the third derivative is bounded in the some neighborhood of λ_0) the scaled reproducing kernel converges uniformly to the sin-kernel. This result for the case of real analytic V was obtained also in [7]. The universality of the reproducing kernel in the bulk for very general conditions on the potential V was proved recently also in [12].

Universality near the edge, i.e., the case when λ_0 is the edge point of the spectrum and $\rho(\lambda) \sim |\lambda - \lambda_0|^{1/2}$, as $\lambda \sim \lambda_0$, was studied in [7]. It was proved that

$$\lim_{n\to\infty} \frac{1}{n^{2/3}\gamma} K_{n,2}(\lambda_0 + s_1/n^{2/3}\gamma, \lambda_0 + s_2/n^{2/3}\gamma) = Q_{\mathrm{Ai}}(s_1, s_2),$$

where

$$Q_{\mathrm{Ai}}(s_1, s_2) = \frac{\mathrm{Ai}(s_1)\mathrm{Ai}'(s_2) - \mathrm{Ai}'(s_1)\mathrm{Ai}(s_2)}{s_1 - s_2} = \int_0^\infty \mathrm{Ai}(s_1 + t)\mathrm{Ai}(s_2 + t)dt. \tag{1.9}$$

This result for GUE ($V(\lambda) = \lambda^2/2$) was obtained in [24]. There are also results on universality near the extreme point, where $\rho(\lambda) \sim (\lambda - \lambda_0)^2$, as $\lambda \sim \lambda_0$ (see [4] for real analytic V and [18] for general V).

For orthogonal ensembles ($\beta = 1$) the situation is more complicated. Instead of (1.6) we need to use the matrix kernel

$$\widehat{K}_n(\lambda, \mu) = \begin{pmatrix} S_n(\lambda, \mu) & S_n d(\lambda, \mu) \\ I S_n(\lambda, \mu) - \epsilon(\lambda - \mu) & S_n(\mu, \lambda) \end{pmatrix}. \tag{1.10}$$

Here

$$S_n(\lambda, \mu) = -\sum_{i,j=0}^{n-1} \psi_i^{(n)}(\lambda)(\mathcal{M}^{(0,n)})^{-1}_{i,j}(n\epsilon\psi_j^{(n)})(\mu), \tag{1.11}$$

where $\psi_i^{(n)}$ are defined by (1.7)–(1.8) and the matrix $\mathcal{M}^{(0,n)}$ is defined as

$$M_{j,l} = n(\psi_j^{(n)}, \epsilon\psi_l^{(n)}); \qquad \mathcal{M}^{(0,\infty)} = \{M_{j,l}\}_{j,l=0}^{\infty}; \qquad \mathcal{M}^{(0,n)} = \{M_{j,l}\}_{j,l=0}^{n-1}, \tag{1.12}$$

where ϵ is the integral operator with the kernel

$$\epsilon(\lambda) = \frac{1}{2}\mathrm{sign}(\lambda); \qquad \epsilon f(\lambda) = \int \epsilon(\lambda - \mu) f(\mu) d\mu. \tag{1.13}$$

The symbol d in (1.10) denotes the differentiating with respect to μ, and $I S_n(\lambda, \mu)$ means the composition of operators ϵ and S_n. Similarly to the hermitian case all marginal densities can be expressed in terms of the kernel $\widehat{K}_n$ (see [23]). In particular, the gap probability has the form

$$E_{n,1}(\Delta_n(a, b)) = \det{}^{1/2}(1 - \widehat{K}_n(\Delta_n(a, b))), \tag{1.14}$$

where $\widehat{K}_n(\Delta_n(a, b))$ is an integral operator from $L^2(\Delta_n(a, b)) \oplus L^2(\Delta_n(a, b))$ to itself defined by the matrix kernel (1.10) and det means it Fredholm determinant. The matrix kernel (1.10) was introduced first in [9] for circular ensemble and then in [13] for orthogonal ensembles. The scalar kernels of (1.10) could be defined in principle in terms of any family of polynomials complete in $L_2(\mathbb{R}, w_n)$ (see [23]), but usually the families of skew orthogonal polynomials were used (see [13] and references therein). Unfortunately, using of skew orthogonal polynomials for general V rises serious technical difficulties.

The main technical obstacle to study the kernel (1.11) defined in terms of orthogonal polynomials is that there is no uniform bound for $\|(\mathcal{M}^{(0,n)})^{-1}\|$. According to Widom (see

[25]), if the potential V is a rational function, then to control $(\mathcal{M}^{(0,n)})^{-1}$ it is enough to control the inverse of some matrix of fixed size depending of V (e.g. if V is polynomial of degree $2m$, then we should control some $(2m-1)\times(2m-1)$ matrix). In the paper [5] by constructing of the exact expressions for the entries of the Widom matrix, it was shown that it is invertible in the case $V(\lambda)=\lambda^{2m}$. This allowed to prove bulk universality for the case $V(\lambda)=\lambda^{2m}+n^{-1/2m}a_{2m-1}\lambda^{2m-1}+\cdots$ (in our notations). The same approach was used in [6] to prove edge universality and in [8] to prove bulk and edge universality (including the case of hard edge) for the Laguerre type ensembles with monomial V. In the papers [20, 21] universality in the bulk and near the edges were studied for V being an even quatric polynomial. The most general result for the moment was obtained in [19], where it was shown that in the one interval case the matrix, which we need to control, is of rank one. This allowed to study any real analytical potential with one interval support and to simplify considerably the proof. In the present paper we will use the result of [19] to prove that up to some small terms (which do not contribute in the limit) the kernel S is the same that for GOE case (see Lemma 1). This allows us to use the method of [24] to prove universality of limiting kernels near the edges. The only difference with [24] is that we use asymptotic of orthogonal polynomials of [7], instead of classical asymptotic of Hermite polynomials.

One more difficulty of real symmetric matrix models appears when we study the gap probability (1.14). It is easy to see that the integral operator with the kernel $\widehat{K}_n(2+x/\gamma n^{2/3},\lambda_0+y/\gamma n^{2/3})$ defined in (1.10) is not a trace class operator in $L^2(s,\infty)\oplus L^2(s,\infty)$ (recall that $A:\mathcal{H}_1\to\mathcal{H}_2$ is a trace class operator if $\|A\|_1:=\mathrm{Tr}(A^*A)^{1/2}<\infty$). To take care of this problem we follow the approach of [24] and use weighted L^2 spaces. If we take any w such that $w^{-1}\in L^1$ and w grows at infinity not faster than exponent, then $\widehat{K}_n$ is a Hilbert-Schmidt operator on $L^2(s,\infty;w)\oplus L^2(s,\infty;w^{-1})$. The diagonal entries of $\widehat{K}_n$ are finite rank, hence trace class. Now the definition of determinant extends to Hilbert-Schmidt operator matrices $\widehat{T}$ with trace class diagonal entries by setting

$$\det(I-\widehat{T})=\det{}_2(I-\widehat{T})e^{-\mathrm{Tr}\,\widehat{T}},$$

where Tr denotes the sum of the traces of the diagonal entries of $\widehat{T}$ and the $\det_2$ is the regularized 2-determinant, defined for the Hilbert-Schmidt operator T with eigenvalues t_k as

$$\det{}_2(I-\widehat{T})=\prod(1-t_k)e^{t_k}$$

(see [10], Sect. IV.2). It follows from the identity for 2-determinant

$$\det{}_2(I-\widehat{T}_1)(I-\widehat{T}_2)e^{\mathrm{Tr}\,\widehat{T}_1\widehat{T}_2}=\det{}_2(I-\widehat{T}_1)\det{}_2(I-\widehat{T}_2)$$

that for this extended definition we still have the relation

$$\det(I-\widehat{T}_1)(I-\widehat{T}_2)=\det(I-\widehat{T}_1)\det(I-\widehat{T}_2).$$

Moreover, if the sequence of the Hilbert-Schmidt operators $\{T_n\}$ on $L^2(s,\infty;w)\oplus L^2(s,\infty;w^{-1})$ with trace class diagonal entries converges to T in the corresponding Hilbert-Schmidt norm, then $\det_2(I-\widehat{T}_n)\to\det_2(I-\widehat{T})$ as $n\to\infty$ (see [10], Sect. IV.2). Hence, to study the limit of the gap probability (1.14) it is enough to prove the convergence of the properly scaled entries of the matrix kernel (1.10) to some limiting kernel in the trace norm.

An important observation which helps to estimate the trace norms of different operators was done in [24]. Consider a rank one kernel $u(x)v(y)$, where $u\in L^2(s,\infty;w_2)$ and $v\in$

$L^2(s,\infty;w_1^{-1})$ and $u\otimes v: L^2(s,\infty;w_1)\to L^2(s,\infty;w_2)$

$$(u\otimes v\,h)(x)=u(x)\int_s^\infty h(y)v(y)dy. \tag{1.15}$$

Then we have

$$\|u\otimes v\|_1\le \|u\|_{L^2(w_2)}\|v\|_{L^2(w_1^{-1})}. \tag{1.16}$$

Here and below we denote by $\|.\|_{L^2(w)}$ the standard norm in $L^2(s,\infty;w)$:

$$\|u\|^2_{L^2(w)}:=\int_s^\infty |u(x)|^2 w(x)dx.$$

Now let us state our main conditions.

C1. The support σ of IDS of the ensemble consists of a single interval:

$$\sigma=[-2,2].$$

C2. $V(z)$ satisfies (1.2) and is an even analytic function in

$$\Omega[d_1,d_2]=\{z:-2-d_1\le \Re z\le 2+d_1,\ |\Im z|\le d_2\},\quad d_1,d_2>0. \tag{1.17}$$

C3. DOS $\rho(\lambda)$ is strictly positive in the internal points $\lambda\in(-2,2)$ and $\rho(\lambda)\sim|\lambda\mp 2|^{1/2}$, as $\lambda\sim\pm 2$.

C4. The function

$$u(\lambda)=2\int \log|\mu-\lambda|\rho(\mu)d\mu - V(\lambda)$$

achieves its maximum if and only if $\lambda\in\sigma$.

Note (see [2]) that under conditions C1–C4 the limiting density of states (DOS) ρ has the form

$$\rho(\lambda)=\frac{1}{2\pi}P(\lambda)\sqrt{4-\lambda^2}\,\mathbf{1}_{|\lambda|<2}, \tag{1.18}$$

where the function P can be represented in the form

$$P(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{V'(z)-V'(2\cos y)}{z-2\cos y}dy. \tag{1.19}$$

If V is a polynomial of $2m$th degree, then it is evident that $P(z)$ is a polynomial of $(2m-2)$th degree, and conditions C3 guarantee that

$$|P(z)|\le C,\quad z\in\Omega[d_1/2,d_2/2],\qquad P(\lambda)\ge\delta>0,\quad \lambda\in[-2,2]. \tag{1.20}$$

We will use also that under conditions C1–C4 the entries of semi infinite Jacoby matrix $\mathcal{J}^{(n)}$, generated by the recursion relations for orthogonal polynomials (1.8)

$$J_{l+1}^{(n)}\psi_{l+1}^{(n)}(\lambda)+q_l^{(n)}\psi_l^{(n)}(\lambda)+J_l^{(n)}\psi_{l-1}^{(n)}(\lambda)=\lambda\psi_l^{(n)}(\lambda),\qquad J_0^{(n)}=0,\quad l=0,\dots \tag{1.21}$$

satisfy the relations (see [1, 2]): $q_l^{(n)} = 0$ and

$$\left| J_{n+k}^{(n)} - 1 - \frac{k}{2nP(2)} \right| \le C\frac{|k|^2 + n^{2/3}}{n^2}, \quad |k| \le 2n^{1/2}, \tag{1.22}$$

where P is defined by (1.19). Here and everywhere below we denote by $C, C_0, C_1, c, \ldots$ positive n-independent constants (different in different formulas).

To formulate our main result we need also to introduce the scaled kernels:

$$\begin{aligned} \mathcal{S}_n(x, y) &= (n^{2/3}\gamma)^{-1} S_n(2 + x/\gamma n^{2/3}, 2 + y/\gamma n^{2/3}), \\ \mathcal{D}_n(x, y) &= (n^{2/3}\gamma)^{-2} D_n(2 + x/\gamma n^{2/3}, 2 + y/\gamma n^{2/3}), \\ \mathcal{I}_n(x, y) &= I_n(2 + x/\gamma n^{2/3}, 2 + y/\gamma n^{2/3}), \\ \mathcal{K}_n(x, y) &= (n^{2/3}\gamma)^{-1} K_n(2 + x/\gamma n^{2/3}, 2 + y/\gamma n^{2/3}), \end{aligned} \tag{1.23}$$

where S_n is defined by (1.11), K_n is defined in (1.6) and $\gamma = P^{2/3}(2)$, with P defined by (1.19). Observe that the determinant in (1.14) is the same if we replace the interval $(2 + s/n^{2/3}, \infty)$ by (s, ∞) and the kernel $\widehat{K}_n$ by

$$\widehat{\mathcal{K}}_n(x, y) = \begin{pmatrix} \mathcal{S}_n(x, y) & \mathcal{D}_n(x, y) \\ \mathcal{I}_n(x, y) - \epsilon(x - y) & \mathcal{S}_n(y, x) \end{pmatrix}. \tag{1.24}$$

Theorem 1 *Consider an orthogonal ensemble of random matrices* (1.3) *with* $\beta = 1$, *and* V *satisfying conditions* C1–C4. *Take* γ *defined in* (1.23). *Then for even* n *we have*:

(i) *if* $p_{l1}^{(n)}$ *is the lth marginal of* (1.3), *then* $n^{l/3}\gamma^{-l} p_{l1}^{(n)}(2 + x_1/\gamma n^{2/3}, \ldots, 2 + x_l/\gamma n^{2/3})$ *converges uniformly in* $x_j \ge s > -\infty$, $j = 1, \ldots, l$ *to the limits coinciding with that for GOE and given in terms of*

$$\widehat{Q}_{\mathrm{Ai}}(x, y) = \lim_{n\to\infty} \widehat{\mathcal{K}}_n(x, y), \tag{1.25}$$

where $\mathcal{K}_n$ *is defined by* (1.23)–(1.24) *and*

$$\widehat{Q}_{\mathrm{Ai}}(x, y) = \begin{pmatrix} S_{\mathrm{Ai}}(x, y) & D_{\mathrm{Ai}}(x, y) \\ I_{\mathrm{Ai}}(x, y) - \epsilon(x - y) & S_{\mathrm{Ai}}(y, x) \end{pmatrix} \tag{1.26}$$

with

$$\begin{aligned} S_{\mathrm{Ai}}(x, y) &= Q_{\mathrm{Ai}}(x, y) + \frac{1}{2}\mathrm{Ai}(x)\left(1 - \int_y^\infty \mathrm{Ai}(z)dz\right), \\ D_{\mathrm{Ai}}(x, y) &= -\partial_y Q_{\mathrm{Ai}}(x, y) - \frac{1}{2}\mathrm{Ai}(x)\mathrm{Ai}(y), \\ I_{\mathrm{Ai}}(x, y) &= -\int_x^\infty Q_{\mathrm{Ai}}(z, y)dz + \frac{1}{2}\left(\int_y^x \mathrm{Ai}(z)dz + \int_x^\infty \mathrm{Ai}(z)dz \int_y^\infty \mathrm{Ai}(z)dz\right), \end{aligned} \tag{1.27}$$

and $Q_{\mathrm{Ai}}(x, y)$ *of* (1.9);

(ii) *if* $E_{n,1}$ *is the gap probability* (1.14) *of* (1.3), *corresponding to the semi-infinite interval* $(2 + s/\gamma n^{2/3}, \infty)$, *then*

$$\lim_{n\to\infty} E_{n,1}\left((2 + s/\gamma n^{2/3}, \infty)\right) := E_1^{(edge)}(s) = \det{}^{1/2}(I - \widehat{Q}_{\mathrm{Ai}}(s)), \tag{1.28}$$

where $\widehat{Q}_{\mathrm{Ai}}(s)$ *is the integral operator, defined in* $L^2(s,\infty;w)\oplus L^2(s,\infty;w^{-1})$ *by the* 2×2 *matrix kernel* (1.26), (1.27) *with* $w(x)=x^2+1$.

We prove Theorem 1 in three steps. On the first step (see Sect. 2.1) we use results of [19] to write the matrix $(\mathcal{M}^{(0,n)})^{-1}$ of (1.12) in the form, which allows to represent the entries of (1.10) in terms of the kernel (1.6) plus the remainders, written in the form convenient for the limiting transition. On the second step (see Sect. 2.1) we show that after scaling (1.23) near the edge point the remainder terms give us the same rank one operators as in the case of COE. Then we represent $\mathcal{S}_n$, $\mathcal{D}_n$, $\mathcal{I}_n$ in the same form as for COE. And on the last step (see Sect. 2.1) we perform the limiting transition in the kernel $\mathcal{S}_n$, $\mathcal{D}_n$, $\mathcal{I}_n$, using the method of [24] and asymptotic of orthogonal polynomials of [7].

2 Proof of Theorem 1

2.1 Representation of $(\mathcal{M}^{(0,n)})^{-1}$

In what follows it is more convenient for us to replace the integration over $\mathbb{R}$ in formulas (1.8), (1.13) and (1.14) by the integration over the interval $[-L,L]$ with $L=2+d_1/2$. Thus we start from the remark:

Remark 1 According to the results of [2] and [15], if we restrict the integration in (1.3) by $|\lambda_i|\le L=2+d_1/2$, consider the polynomials $\{p_k^{(n,L)}\}_{k=0}^{\infty}$ orthogonal on the interval $[-L,L]$ with the weight e^{-nV} and set $\psi_k^{(n,L)}=e^{-nV/2}p_k^{(n,L)}$, then for $k\le n(1+\varepsilon)$ with some $\varepsilon>0$

$$\sup_{|\lambda|\le L}|\psi_k^{(n,L)}(\lambda)-\psi_k^{(n)}(\lambda)|\le e^{-nC},\qquad |\psi_k^{(n)}(\pm L)|\le e^{-nC} \tag{2.1}$$

with some absolute C. Therefore from the very beginning we can take all integrals in (1.3), (1.8), (1.13) and (1.12) over the interval $[-L,L]$. Note also that since V is an analytic function in $\Omega[d_1,d_2]$ (see (1.17)), for any $m\in\mathbb{N}$ there exists a polynomial V_m of the $(2m)$th degree such that

$$|V_m(z)|\le C_0,\quad |V(z)-V_m(z)|\le e^{-Cm},\quad z\in\Omega[d_1/2,d_2/2]. \tag{2.2}$$

Take

$$m=[\log^2 n] \tag{2.3}$$

and consider the system of polynomials $\{p_k^{(n,L,m)}\}_{k=0}^{\infty}$ orthogonal in the interval $[-L,L]$ with respect to the weight $e^{-nV_m(\lambda)}$. Set $\psi_k^{(n,L,m)}=p_k^{(n,L,m)}e^{-nV_m/2}$ and construct $\mathcal{M}_m^{(0,n)}$ by (1.12) with $\psi_k^{(n,L,m)}$. Then for any $k\le n+2n^{1/2}$ and uniformly in $\lambda\in[-L,L]$

$$\begin{aligned}&|\psi_k^{(n,L)}(\lambda)-\psi_k^{(n,L,m)}(\lambda)|\le e^{-C\log^2 n},\qquad |\varepsilon\psi_k^{(n,L)}(\lambda)-\varepsilon\psi_k^{(n,L,m)}(\lambda)|\le e^{-C\log^2 n}\\ &\|\mathcal{M}_m^{(0,n)}-\mathcal{M}^{(0,n)}\|\le e^{-C\log^2 n},\qquad \|(\mathcal{M}_m^{(0,n)})^{-1}-(\mathcal{M}^{(0,n)})^{-1}\|\le e^{-C\log^2 n}.\end{aligned} \tag{2.4}$$

The proof of the first bound here is identical to the proof of (2.1) (see [15]). The second bound follows from the first one because the operator $\varepsilon: L_2[-L,L]\to C[-L,L]$ is bounded by L. The third bound in (2.4) follows from the first, and the last bound follows from the third one and from the fact that $\|(\mathcal{M}_m^{(0,n)})^{-1}\|$ is uniformly bounded (see [19]). Hence

$$|S_{n,m}(\lambda,\mu)-S_n(\lambda,\mu)|\le Cn^4e^{-C\log^2 n}\le e^{-C'\log^2 n}, \tag{2.5}$$

and below we will study $S_{n,m}(\lambda,\mu)$ instead of $S_n(\lambda,\mu)$. To simplify notations we omit the indexes m, L, but keep the dependence on m in the estimates. For more detail of the replacement see [19].

To formulate the result of [19] (see Corollary 1) on the representation of $(\mathcal{M}^{(0,n)})^{-1}$ we need also to introduce a few Toeplitz matrices. Consider the infinite matrix $\mathcal{P}=\{P_{j,k}\}_{j,k=-\infty}^{\infty}$ in $l_2[-\infty,\infty]$ with entries

$$P_{j,k}=\frac{1}{2\pi}\int_{-\pi}^{\pi}P(2\cos y)e^{i(j-k)y}dy, \tag{2.6}$$

and $\mathcal{R}=\mathcal{P}^{-1}$,

$$\mathcal{R}^{(0,n)}=\{R_{j,k}\}_{j,k=0}^{n-1},\quad R_{j,k}=R_{j-k}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i(j-k)x}dx}{P(2\cos x)}. \tag{2.7}$$

It is important that (see [19], Proposition 1)

$$|R_{j,k}|\le e^{-c|j-k|},\qquad |((\mathcal{R}^{(0,n)}))^{-1}_{j,k}|\le e^{-c|j-k|}, \tag{2.8}$$

where $c>0$ is some n-independent constant. Remark also that if we denote by $\mathcal{J}^*$ an infinite Jacobi matrix with constant coefficients

$$\mathcal{J}^*=\{J^*_{j,k}\}_{j,k=-\infty}^{\infty},\quad J^*_{j,k}=\delta_{j+1,k}+\delta_{j-1,k}, \tag{2.9}$$

then the spectral theorem yields that $\mathcal{P}=P(\mathcal{J}^*)$, $\mathcal{R}=P^{-1}(\mathcal{J}^*)$.

Two more matrices which we use below have the form

$$\mathcal{D}^{(0,n)}=\{D_{j,k}\}_{j,k=0}^{n-1},\quad D_{j,k}=\delta_{j+1,k}-\delta_{j-1,k} \tag{2.10}$$

and $\mathcal{V}^{(0,\infty)}=\{\mathcal{V}_{j,l}\}_{j,l=0}^{\infty}$, where

$$\mathcal{V}_{j,l}=\operatorname{sign}(l-j)(\psi_j^{(n)},V'\psi_l^{(n)})_2=\frac{2}{n}\begin{cases}(\psi_j^{(n)},(\psi_l^{(n)})')_2, & j>l,\\ (\psi_j^{(n)},(\psi_l^{(n)})')_2+O(e^{-C\log^2 n}), & j\le l.\end{cases} \tag{2.11}$$

Here $O(e^{-C\log^2 n})$ appears because of the integration by parts and bounds (2.1), (2.4).

According to Corollary 1 from [19], under conditions C1–C4

$$(\mathcal{M}^{(0,n)})^{-1}_{j,k}=\mathcal{Q}^{(0,n)}_{j,k}+\frac{1}{2}a_jb_k+O(n^{-1/2}\log^6 n), \tag{2.12}$$

where

$$\mathcal{Q}^{(0,n)}_{j,k}=\frac{1}{2}\begin{cases}\mathcal{V}^{(0,\infty)}_{j,k}, & \text{for } 0\le j\le n-2m,\ 0\le k<n,\\ ((\mathcal{R}^{(0,n)})^{-1}\mathcal{D}^{(0,n)})_{j,k}, & \text{for } n-2m<j<n,\ 0\le k<n,\end{cases} \tag{2.13}$$

and

$$a_j = ((\mathcal{R}^{(0,n)})^{-1}e_{n-1})_j, \qquad b_k = ((\mathcal{R}^{(0,n)})^{-1}r^*)_k, \qquad r^*_{n-i} = R_i \tag{2.14}$$

with R_i defined by (2.7).

Note that since $(\mathcal{R})^{-1}_{j,k} = \mathcal{P}_{j,k} = 0$ for $|j-k| > 2m-2$, the standard linear algebra yields that $(\mathcal{R}^{(0,n)})^{-1}$ possesses the same property, i.e.,

$$(\mathcal{R}^{(0,n)})^{-1}_{j,k} = 0, \text{ for } |j-k| > 2m-2 \Rightarrow \mathcal{Q}^{(0,n)}_{j,k} = 0, \text{ for } |j-k| > 2m-2. \tag{2.15}$$

2.2 Representation of the Scaled Kernel $\widehat{\mathcal{K}_n}$ Near the Edge Point

Lemma 1 *If we denote*

$$\varphi_n(x) = n^{-1/6}\gamma^{-1/2}\psi_n^{(n)}(2 + x/\gamma n^{2/3}), \qquad \psi_n(x) = n^{-1/6}\gamma^{-1/2}\psi_{n-1}^{(n)}(2 + x/\gamma n^{2/3}), \tag{2.16}$$

then

$$\begin{aligned} \mathcal{S}_n(x,y) &= \mathcal{K}_n(x,y) + \frac{1}{2}\psi_n(x)\epsilon\varphi_n(y) + r_n(x,y), \\ \mathcal{D}_n(x,y) &= -\frac{\partial}{\partial y}\mathcal{K}_n(x,y) - \frac{1}{2}\psi_n(x)\varphi_n(y) - \frac{\partial}{\partial y} r_n(x,y), \\ \mathcal{I}_n(x,y) &= \mathcal{I}\mathcal{K}_n(x,y) + \frac{1}{2}\epsilon\psi_n(x)\epsilon\varphi_n(y) + (\epsilon r_n)(x,y), \end{aligned} \tag{2.17}$$

where

$$\|r_n(x,y)\|_1, \left\|\frac{\partial}{\partial y} r_n(x,y)\right\|_1, \|(\epsilon r_n)(x,y)\|_1 \le Cn^{-1/3}\log^6 n. \tag{2.18}$$

Proof Since (2.12), (2.13) and (2.11) imply for $j \le n-2m$

$$n\sum_{k=0}^{n-1}(\mathcal{M}^{(0,n)})^{-1}_{j,k}\epsilon\psi_k^{(n)}(\mu) = -\psi_j^{(n)}(\mu) + O(e^{-c\log^2 n}),$$

we have

$$-n\sum_{j=0}^{n-2m}\sum_{k=0}^{n-1}\psi_j^{(n)}(\lambda)(\mathcal{M}^{(0,n)})^{-1}_{j,k}\epsilon\psi_k^{(n)}(\mu) = \sum_{j=0}^{n-2m}\psi_j^{(n)}(\lambda)\psi_j^{(n)}(\mu) + O(e^{-c\log^2 n}). \tag{2.19}$$

For $n-1 \le j > n-2m$ we need to use the result of [19] (see (68)), according to which for any $|p-n| \le 4\log^2 n$ we have

$$\begin{aligned} &-\frac{1}{2}\left(\epsilon\psi_{p+1}^{(n)}(\mu) - \epsilon\psi_{p-1}^{(n)}(\mu)\right) \\ &\quad = n^{-1}\sum_{l=0}^{\infty}R_{p,l}\psi_l^{(n)}(\mu) + n^{-1}e_p(\mu) \\ &\quad = n^{-1}\sum_{l=0}^{n-1}R_{p,l}^{(0,n)}\psi_l^{(n)}(\mu) + n^{-1}\sum_{l=n}^{\infty}R_{p,l}\psi_l^{(n)}(\mu) + n^{-1}e_p(\mu), \end{aligned} \tag{2.20}$$

where the remainder terms $e_p(\mu)$ satisfy the bounds

$$\|e_p\|_{L^2[-L,L]} \le Cn^{-1/2}\log^4 n.$$

Therefore the scaled functions $\widetilde{e}_p(x) = n^{-1/3}e_p(2+x/\gamma n^{2/3})$ admit the bounds

$$\|\widetilde{e}_p\|_{L^2(w^{-1})} \le Cn^{-1/2}\log^4 n.$$

Hence, using the definition of $\mathcal{D}^{(0,n)}$ (2.10) and (2.20), we obtain

$$\begin{aligned}
-n&\sum_{j=n-2m+1}^{n-1}\sum_{k=0}^{n-1}\psi_j^{(n)}(\lambda)\left((\mathcal{R}^{(0,n)})^{-1}\mathcal{D}^{(0,n)}\right)_{j,k}\epsilon\psi_k^{(n)}(\mu)\\
&=\sum_{j=n-2m+1}^{n-1}\psi_j^{(n)}(\lambda)\psi_j^{(n)}(\mu)\\
&\quad+\frac{n}{2}\epsilon\psi_n^{(n)}(\mu)\sum_{j=n-2m+1}^{n-1}\psi_j^{(n)}(\lambda)(\mathcal{R}^{(0,n)})^{-1}_{j,n-1}+r_n^{(1)}(\lambda,\mu)+r_n^{(2)}(\lambda,\mu),
\end{aligned}\tag{2.21}$$

where $r_n^{(1)}(\lambda,\mu)$ collects the terms, which appear because of the second sum in the r.h.s. of (2.20), and $r_n^{(2)}(\lambda,\mu)$ collects the remainder terms e_p of (2.20):

$$\begin{aligned}
r_n^{(1)}(\lambda,\mu) &:= \sum_{j=n-2m+1}^{n-1}\sum_{p=0}^{n-1}\sum_{l=n}^{\infty}(\mathcal{R}^{(0,n)})^{-1}_{j,p}R_{p,l}\psi_j^{(n)}(\lambda)\psi_l^{(n)}(\mu),\\
r_n^{(2)}(\lambda,\mu) &:= \sum_{j=n-2m+1}^{n-1}\sum_{p=0}^{n-1}(\mathcal{R}^{(0,n)})^{-1}_{j,p}\psi_j^{(n)}(\lambda)e_p(\mu).
\end{aligned}$$

□

Definition 1 We will say that some remainder kernel $r_n^{(\alpha)}(\lambda,\mu)$ $(\alpha = 1,2,\dots)$ satisfies the bound B with exponents κ_1, κ_2 and κ_3, if

$$\|n^{-2/3}r_n^{(\alpha)}(2+x/\gamma n^{2/3}, 2+y/\gamma n^{2/3})\|_1 \le Cn^{-\kappa_1}m^{\kappa_2}\log^{\kappa_3} n.\tag{B}$$

According to the results of [7], we have

$$\begin{aligned}
n^{-1/6}\gamma^{-1/2}\psi_j^{(n)}(2+x/\gamma n^{2/3}) &= \mathrm{Ai}(x+(n-j)/c_* n^{1/3})(1+O(n^{-1/3})),\\
|n^{-1/6}\gamma^{-1/2}\psi_j^{(n)}(2+x/\gamma n^{2/3})| &\le Ce^{-x},
\end{aligned}\tag{2.22}$$

where c_* is some constant not important for us. These asymptotic implies, in particular, that

$$\begin{aligned}
\|n^{-1/6}\gamma^{-1/2}\psi_j^{(n)}(2+x/\gamma n^{2/3})\|_{L^2(w)} &\le C,\\
\|n^{-1/6}\gamma^{-1/2}\psi_j^{(n)}(2+x/\gamma n^{2/3})\|_{L^2(w^{-1})} &\le C.
\end{aligned}\tag{2.23}$$

Using the asymptotic, (2.8) and (1.16), we obtain that $r_n^{(1)}(\lambda,\mu)$ satisfies (B) with $\kappa_1 = 1/3$, $\kappa_2 = 1$, $\kappa_3 = 0$.

Similarly, using (2.20), (2.22), (2.8) and (1.16), we get that $r_n^{(2)}(\lambda,\mu)$ satisfies (B) with $\kappa_1=2/3$, $\kappa_2=1$, $\kappa_3=4$.

Moreover, if we denote

$$r_n^{(3)}(\lambda,\mu)=n\epsilon\psi_n^{(n)}(\mu)\sum_{j=n-2m+1}^{n-1}(\psi_j^{(n)}(\lambda)-\psi_{n-1}^{(n)}(\lambda))(\mathcal{R}^{(0,n)})^{-1}_{j,n-1} \tag{2.24}$$

then (2.19) and (2.21) give us

$$\begin{aligned}&-n\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\psi_j^{(n)}(\lambda)(\mathcal{M}^{(0,n)})^{-1}_{j,k}\epsilon\psi_k^{(n)}(\mu)\\&\quad=K_n(\lambda,\mu)+\frac{n}{2}\epsilon\psi_n^{(n)}(\mu)\psi_{n-1}^{(n)}(\lambda)((\mathcal{R}^{(0,n)})^{-1}e_{n-1},u)+r_n^{(1)}(\lambda,\mu)\\&\qquad+r_n^{(2)}(\lambda,\mu)+r_n^{(3)}(\lambda,\mu),\end{aligned} \tag{2.25}$$

where u is a vector, whose components are given by

$$u_i=1,\quad i\in[n-2m,n-1],\qquad u_i=0,\quad i\notin[n-2m,n-1], \tag{2.26}$$

and the remainder term $r_n^{(3)}$ in view of (2.22), (2.8) and (1.16) satisfies the bound (B) with $\kappa_1=1/3$, $\kappa_2=1$, $\kappa_3=0$.

Now we consider the term (see (2.12))

$$A(\lambda,\mu)=\frac{n}{2}\sum_{k=n-2m}^{n-1}a_k\psi_k^{(n)}(\lambda)\sum_{j=n-2m}^{n-1}b_j\epsilon\psi_j^{(n)}(\mu).$$

Using (2.20) and (2.22), similarly to the above it is easy to obtain that

$$A(\lambda,\mu)=\frac{n}{2}(a,u)(b,u)\,\psi_{n-1}^{(n)}(\lambda)\,\epsilon\psi_n^{(n)}(\mu)+r_n^{(4)}(\lambda,\mu), \tag{2.27}$$

where u is defined in (2.26), and the remainder $r_n^{(4)}$ satisfies the bound (B) with $\kappa_1=1/3$, $\kappa_2=0$, $\kappa_3=0$.

Let us find $(a,u)(b,u)$. Making transposition in (2.12) and taking into account that $(\mathcal{M}^{(0,n)})^{-1}$ and $\mathcal{D}^{(0,n)}$ are skew symmetric matrices, we get

$$-(\mathcal{M}^{(0,n)})^{-1}_{j,k}=-\frac{1}{2}(\mathcal{D}^{(0,n)}(\mathcal{R}^{(0,n)})^{-1})_{j,k}+\frac{1}{2}a_kb_j+O(n^{-1/2}\log n).$$

Taking the sum of the equation with (2.12) and applying the result to u we get

$$(a,u)(b,u)=\frac{1}{2}([\mathcal{D}^{(0,n)},(\mathcal{R}^{(0,n)})^{-1}]u,u)=-((\mathcal{R}^{(0,n)})^{-1}u,\mathcal{D}^{(0,n)}u)+O(n^{-1/2}m^2\log n).$$

But it is easy to see that

$$\mathcal{D}^{(0,n)}u=-e_{n-1}+e_{n-2m}+e_{n-2m-1}.$$

Hence,

$$(a,u)(b,u)=((\mathcal{R}^{(0,n)})^{-1}e_{n-1},u)-((\mathcal{R}^{(0,n)})^{-1}(e_{n-2m}+e_{n-2m-1}),u)+O(n^{-1/2}m^2\log n).$$

Moreover, since $\mathcal{P}=\mathcal{R}^{-1}$ has only $2m-2$ nonzero diagonals, the standard linear algebra argument yields that for $j\le n-2m$ $(\mathcal{R}^{(0,n)})^{-1}e_j=\mathcal{P}e_j$. Then, using (2.6) and (2.26), we obtain

$$(\mathcal{P}(e_{n-2m}+e_{n-2m-1}),u)=P(2).$$

Finally

$$(a,u)(b,u)=((\mathcal{R}^{(0,n)})^{-1}e_{n-1},u)-P(2)+O(n^{-1/2}m^2\log n). \tag{2.28}$$

Then, combining (2.25) with (2.28) and bounds (B) for $r_n^{(\alpha)}(\lambda,\mu)$ with $\alpha=1,2,3,4$, we get the first line of (2.17). The second line of (2.17) can be proved similarly, if we use that (2.22) can be differentiated. To prove the last line of (2.17) we used that (2.22) implies that for $|k-n|=o(n)$

$$|\epsilon\psi_k(\mu)|\le Cn^{-1/2}. \tag{2.29}$$

Hence

$$\|\epsilon\psi_k(2+x/\gamma n^{2/3})\|_{L^2(w^{-1})}\le Cn^{-1/2}. \tag{2.30}$$

Using these bounds we get the estimates for $\epsilon r^{(1)}(\lambda,\mu)$ and $\epsilon r^{(2)}(\lambda,\mu)$. The bound for $\epsilon r^{(3)}(\lambda,\mu)$ and $\epsilon r^{(4)}(\lambda,\mu)$ follow from (2.20), (1.16), (2.23) and (2.8). □

Let us transform the kernel K_n. We use the representation

$$K_n(\lambda,\mu)=\frac{n}{2}\sum_{k=0}^{n-1}\sum_{j=n}^{\infty}V'(\mathcal{J}^{(n)})_{j,k}\times\int_0^\infty d\nu\left(\psi_k^{(n)}(\lambda+\nu)\psi_j^{(n)}(\mu+\nu)+\psi_k^{(n)}(\mu+\nu)\psi_j^{(n)}(\lambda+\nu)\right). \tag{2.31}$$

The representation can be obtained by taking $\frac{\partial}{\partial\lambda}+\frac{\partial}{\partial\mu}$ from both sides of (2.31) and using of (2.11) to expand $\frac{\partial}{\partial\lambda}\psi_k^{(n)}(\lambda)$ with respect to the basis $\{\psi_j^{(n)}\}_{j=1}^\infty$.

Using the same trick as above, on the basis of (2.22) and (1.16) it is easy to show that

$$K_n(\lambda,\mu)=\frac{n}{2}\left(\sum_{k=0}^{n-1}\sum_{j=n}^{\infty}V'(\mathcal{J}^{(n)})_{j,k}\right)\int_0^\infty d\nu\left(\psi_n^{(n)}(\lambda+\nu)\psi_{n-1}^{(n)}(\mu+\nu)+\psi_n^{(n)}(\mu+\nu)\psi_{n-1}^{(n)}(\lambda+\nu)\right)+r_n^{(5)}(\lambda,\mu), \tag{2.32}$$

where $r_n^{(5)}(\lambda,\mu)$ satisfies (B) with $\kappa_1=1/3$, $\kappa_2=\kappa_3=0$.

Moreover, using (1.22), we have

$$V_n^s:=\sum_{k=0}^{n-1}\sum_{j=n}^{\infty}V'(\mathcal{J}^{(n)})_{j,k}=\sum_{k=0}^{n-1}\sum_{j=n}^{\infty}V'(\mathcal{J}^{*})_{j,k}+O(n^{-1})=\sum_{k=1}^{\infty}kV_k'+O(n^{-1}),$$

where

$$V_k'=\frac{1}{2\pi}\int_{-\pi}^{\pi}V'(2\cos x)e^{ikx}dx.$$

On the other hand, it is evident that if we consider

$$\mathcal{V}(x) = \sum_{k=0}^{n-1} V_k' \sin kx,$$

then

$$V_n^s = \frac{d}{dx}\mathcal{V}(x)\bigg|_{x=0} + O(n^{-1}).$$

But it was proved in [19] (see Lemma 1) that $\mathcal{V}(x) = \sin x\, P(2\cos x)$. Thus we get

$$V_n^s = P(2) + O(n^{-1}),$$

and we obtain from (2.32) that the kernel $\mathcal{K}_n$ from (1.23) can be represented in the form

$$\mathcal{K}_n(x,y) = \frac{1}{2}\int_0^\infty dz\,(\psi_n(x+z)\varphi_n(y+z) + \psi_n(y+z)\varphi_n(x+z)) + r_n^{(6)}(x,y), \tag{2.33}$$

where $\|r_n^{(6)}\|_1 \le Cn^{-1/3}$. Hence, the kernel $\mathcal{S}_n$ is represented in the form

$$\begin{aligned}\mathcal{S}_n(x,y) = {}& \frac{1}{2}\int_0^\infty dz\,(\psi_n(x+z)\varphi_n(y+z) + \psi_n(y+z)\varphi_n(x+z)) \\ & + \frac{n}{2}\psi_n(x)\epsilon\varphi_n(y) + r_n(x,y), \quad \|r_n\|_1 \le Cmn^{-1/3}.\end{aligned} \tag{2.34}$$

Corresponding representations for $\mathcal{D}_n$ and $\mathcal{I}_n$ follows from this one.

2.3 Limiting Transition in the Kernels

We will prove now that the kernels $\mathcal{S}_n$, $\mathcal{D}_n$ and $\mathcal{I}_n$ converge to their limit in the corresponding trace norms. As it was mentioned in Introduction, it is enough to show the convergence of $\det(I - \widehat{\mathcal{K}_n})$ to $\det(I - \widehat{\mathcal{Q}_{\mathrm{Ai}}})$, i.e., to prove assertion (ii) of Theorem 1.

Using (2.34), we can prove that $\mathcal{S}_n$ converges in the trace norm to S_{Ai}, repeating almost literally argument of [24], but using (2.22) instead of classical asymptotic for Hermite polynomials. Indeed, relations (2.22) yield

$$\lim_{n\to\infty} \|\varphi_n(\cdot+z) - \mathrm{Ai}(\cdot+z)\|_{L^2(w)} = \lim_{n\to\infty} \|\psi_n(\cdot+z) - \mathrm{Ai}(\cdot+z)\|_{L^2(w)} = 0. \tag{2.35}$$

Let us prove that $\mathcal{K}_n : L^2(w) \to L^2(w)$ of (1.23) converges in the $\|\ldots\|_1$ norm to $Q_{\mathrm{Ai}} : L^2(w) \to L^2(w)$, where Q_{Ai} is defined in (1.9). Using (2.22), (1.9), and (1.16) we have

$$\begin{aligned}&\|\mathcal{K}_n - Q_{\mathrm{Ai}}\|_1 \\ &\quad \le \frac{1}{2}\int_0^\infty \big(\|\varphi_n(\cdot+z) - \mathrm{Ai}(\cdot+z)\|_{L^2(w)} + \|\psi_n(\cdot+z) - \mathrm{Ai}(\cdot+z)\|_{L^2(w)}\big) \\ &\qquad \times \big(\|\varphi_n(\cdot+z)\|_{L^2(w^{-1})} + \|\psi_n(\cdot+z)\|_{L^2(w^{-1})} + \|\mathrm{Ai}(\cdot+z)\|_{L^2(w^{-1})}\big)\,dz \\ &\quad \le C\int_0^\infty \big(\|\varphi(\cdot+z) - \mathrm{Ai}(\cdot+z)\|_{L^2(w)} \\ &\qquad + \|\psi(\cdot+z) - \mathrm{Ai}(\cdot+z)\|_{L^2(w)}\big)e^{-z}dz\end{aligned} \tag{2.36}$$

for some n-independent $C > 0$. Here we have again used (2.22), implying

$$\|\varphi_n(\cdot + z)\|_{L^2(w^{-1})} \le Ce^{-z}, \qquad \|\psi_n(\cdot + z)\|_{L^2(w^{-1})} \le Ce^{-z}. \tag{2.37}$$

Now we can use the dominated convergence theorem to make the limit $n \to \infty$ in (2.36).

To pass to the limit $n \to \infty$ in $\psi_n(x)\epsilon\varphi_n(y)$, remark that uniformly in $y > s$

$$\begin{aligned} \epsilon\varphi_n(y) &= c_{\varphi_n} - \int_y^{n^{2/3}\gamma(L-2)} \varphi_n(z)dz, \\ c_{\varphi_n} &= \frac{1}{2}\int_{-n^{2/3}\gamma(L+2)}^{n^{2/3}\gamma(L-2)} \varphi_n(z)dz = \frac{n^{1/2}}{2}\int_{-L}^{L} \psi_n^{(n)}(\lambda)d\lambda, \end{aligned} \tag{2.38}$$

where L was defined in Remark 1. But according to the results of [7],

$$\begin{aligned} &n^{1/2}\int_{-2+n^{-1/4}}^{2-n^{-1/4}} \psi_n^{(n)}(\lambda)d\lambda \to 0, \\ &n^{1/2}\left(\int_{2+n^{-1/4}}^{L} + \int_{-L}^{-2-n^{-1/4}}\right)\psi_n^{(n)}(\lambda)d\lambda \to 0, \quad n \to \infty. \end{aligned}$$

Thus, (2.22) and the evenness of $\psi_n^{(n)}$ yield

$$\lim_{n\to\infty} c_{\varphi_n} = \lim_{n\to\infty} n^{1/2}\int_{2-n^{-1/4}}^{2+n^{-1/4}} \psi_n^{(n)}(\lambda)d\lambda = \int_{-\infty}^{\infty} \mathrm{Ai}(x)dx = 1.$$

Moreover, (2.22) allow us to pass to the limit $n \to \infty$ in the second term of the representation (2.38) of $\epsilon\varphi_n$. Thus we have uniformly in $y \ge s > -\infty$

$$\lim_{n\to\infty} \epsilon\varphi_n(y) = 1 - \int_y^{\infty} \mathrm{Ai}(z)dz.$$

Now (1.16) implies

$$\lim_{n\to\infty}\left(\mathcal{K}_n(x,y) + \frac{1}{2}\psi_n(x)\epsilon\varphi_n(y)\right) = Q_{\mathrm{Ai}}(x,y) + \frac{1}{2}\mathrm{Ai}(x)\left(1 - \int_y^{\infty}\mathrm{Ai}(z)dz\right),$$

where the limit is understood in the $\|\ldots\|_1$-norm.

To prove that $-\partial_y\mathcal{K}_n : L^2(w^{-1}) \to L^2(w)$ converges in the $\|\ldots\|_1$ norm to $-\partial_y Q_{\mathrm{Ai}} : L^2(w^{-1}) \to L^2(w)$, we repeat the argument used in (2.36), taking into account (1.16) with $w_1 = w^{-1}$ and $w_2 = w$. Moreover, we use the relations (2.35),

$$\|\varphi_n(\cdot + z)\|_{L^2(w)} \le C'e^{-z}, \qquad \|\psi_n(\cdot + z)\|_{L^2(w)} \le C'e^{-z}, \tag{2.39}$$

and

$$\lim_{n\to\infty}\|\varphi_n'(\cdot + z) - \mathrm{Ai}'(\cdot + z)\|_{L^2(w)} = \lim_{n\to\infty}\|\psi_n'(\cdot + z) - \mathrm{Ai}'(\cdot + z)\|_{L^2(w)} = 0.$$

We obtain then

$$\lim_{n\to\infty}\left(-\partial_y\mathcal{K}_n(x,y)-\psi_n(x)\varphi_n(y)\right)=-\partial_y Q_{\mathrm{Ai}}(x,y)-\frac{1}{2}\mathrm{Ai}(x)\mathrm{Ai}(y),$$

where the limit is understood in the $\|\dots\|_1$ norm.

We are left to prove that $\mathcal{I}_n : L^2(w)\to L^2(w^{-1})$ converges in the $\|\dots\|_1$ norm to the operator from $L^2(w)$ to $L^2(w^{-1})$, defined by the kernel

$$\int \epsilon(x-x')Q_{\mathrm{Ai}}(x',y)dx'.$$

To this end denote

$$\Phi_n(x)=\int_x^\infty \varphi_n(x')dx', \qquad \Psi_n(x)=\int_x^\infty \psi_n(x')dx'.$$

Then

$$\begin{aligned}
\epsilon\varphi_n(x)&=\frac{1}{2}\int_{-\infty}^{\infty}\varphi_n(x')dx'-\Phi_n(x)=c_{\varphi_n}-\Phi_n(x),\\
\epsilon\psi_n(x)&=\frac{1}{2}\int_{-\infty}^{\infty}\psi_n(x')dx'-\Psi_n(x)=-\Psi_n(x)\\
\epsilon\mathcal{K}_n(x,y)&=\frac{1}{2}\int_0^\infty dz(\epsilon\psi_n(x+z)\varphi_n(y+z)+\epsilon\varphi_n(x+z)\psi_n(y+z))\\
&=-\frac{1}{2}\int_0^\infty dz(\Psi_n(x+z)\varphi_n(y+z)+\Phi_n(x+z)\psi_n(y+z))+\frac{c_{\varphi_n}}{2}\Psi_n(y).
\end{aligned}$$

Here the second relation follows from the fact that ψ_n is an odd function (because $\psi_{n-1}^{(n)}$ is odd), and the third one follows from the first, and the second, combined with (2.38). Hence, repeating again the argument used in (2.36) and taking into account that (2.22) implies

$$\lim_{n\to\infty}\left\|\Psi_n(x)-\int_x^\infty \mathrm{Ai}(x')dx'\right\|_{L^2(w^{-1})}=\lim_{n\to\infty}\left\|\Phi_n(x)-\int_x^\infty \mathrm{Ai}(x')dx'\right\|_{L^2(w^{-1})}=0,$$

and

$$\|\Psi_n(\cdot+z)\|_{L^2(w^{-1})}\le Ce^{-z}, \qquad \|\Phi_n(\cdot+z)\|_{L^2(w^{-1})}\le Ce^{-z},$$

we obtain

$$\begin{aligned}
\lim_{n\to\infty}\mathcal{I}_n(x,y)=&-\int_x^\infty Q_{\mathrm{Ai}}(x',y)dx'\\
&+\frac{1}{2}\left(\int_y^x \mathrm{Ai}(x')dx'+\int_x^\infty \mathrm{Ai}(x')dx'\int_y^\infty \mathrm{Ai}(y')dy'\right),
\end{aligned}$$

where the limit is understood in the $\|\dots\|_1$ norm. Thus we have proved assertion (ii).

Note that we have also proved that $\lim_{n\to\infty}\widehat{\mathcal{K}}_n(x,y)=\widehat{Q}_{\mathrm{Ai}}(x,y)$ uniformly in $x,y\in(s,\infty)$. Hence assertion (i) is also proved. □

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